

Canonical equations of Hamilton with beautiful symmetry

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Abstract

The Hamiltonian formulation plays the essential role in constructing the framework of modern physics. In this paper, a new form of canonical equations of Hamilton with the complete symmetry is obtained, which are valid not only for the first-order differential system, but also for the second-order differential system. The conventional form of the canonical equations without the symmetry [Goldstein *et al.*, Classical Mechanics, 3rd ed, Addison-Wesley, 2001] are only for the second-order differential system. It is pointed out for the first time that the number of the canonical equations for the first-order differential system is half of that for the second-order differential system. The nonlinear Schrödinger equation, a universal first-order differential system, can be expressed with the new canonical equations in a consistent way.

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The Hamiltonian viewpoint provides a framework for theoretical extensions in many areas of physics. In classical mechanics it forms the basis for further developments, such as Hamilton-Jacobi theory, perturbation approaches and chaos [1]. The canonical equations of Hamilton in classical mechanics are expressed as [1]

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad (i = 1, \dots, n), \quad (1)$$

$$-\dot{p}_i = \frac{\partial H}{\partial q_i}, \quad (i = 1, \dots, n), \quad (2)$$

where q_i and p_i are said to be the generalized coordinate and the generalized momentum, $\dot{q}_i = dq_i/dt$ is the generalized velocity, $\dot{p}_i = dp_i/dt$, and H is the Hamiltonian of the system. The generalized momentum p_i is defined as $p_i = \frac{\partial L}{\partial \dot{q}_i}$, where L is the Lagrangian of the system. And the Hamiltonian H is obtained by the Legendre transformation $H = \sum_{i=1}^n \dot{q}_i p_i - L$. A set of (q_i, p_i) forms a $2n$ -dimensional phase space.

The canonical equations (1) and (2) are not only valid for the discrete system, but also can be extended to the continuous system as [1]

$$\dot{q}_s = \frac{\delta h}{\delta p_s}, \quad (s = 1, \dots, N), \quad (3)$$

$$-\dot{p}_s = \frac{\delta h}{\delta q_s}, \quad (s = 1, \dots, N), \quad (4)$$

where the subscript s represents the components of the quantity of the continuous system [1], hereafter it all denotes $1, \dots, N$, $\frac{\delta h}{\delta q_s} = \frac{\partial h}{\partial q_s} - \frac{\partial}{\partial x} \frac{\partial h}{\partial q_{s,x}}$ and $\frac{\delta h}{\delta p_s} = \frac{\partial h}{\partial p_s} - \frac{\partial}{\partial x} \frac{\partial h}{\partial p_{s,x}}$ denote the functional derivatives of h with respect to q_s and p_s with $q_{s,x} = \frac{\partial q_s}{\partial x}$ and $p_{s,x} = \frac{\partial p_s}{\partial x}$, q_s and p_s are the generalized coordinate and the generalized momentum, respectively, and h is the Hamiltonian density of the continuous system. Like the discrete system, the generalized momentum p_s for the continuous system is defined as

$$p_s = \frac{\partial l}{\partial \dot{q}_s}, \quad (5)$$

and the Hamiltonian density h for the continuous system is obtained by the Legendre transformation as

$$h = \sum_{s=1}^N \dot{q}_s p_s - l, \quad (6)$$

where l is the Lagrangian density. But it is significantly different for the continuous system that q_s and p_s are now not only functions of time t , but also the spatial coordinate x , where

the spatial coordinate x is not the generalized coordinate, but only serves as the continuous index replacing the discrete i in Eqs.(1) and (2). To distinguish t from the coordinate x , we refer time t as the evolution coordinate. p_s and q_s define the infinite-dimensional phase space. h is a function of q_s, p_s and $q_{s,x}$ but not $p_{s,x}$, so $\frac{\delta h}{\delta p_s} = \frac{\partial h}{\partial p_s}$, then Eq.(3) can be also expressed as

$$\dot{q}_s = \frac{\partial h}{\partial p_s}. \quad (7)$$

“The advantages of the Hamiltonian formulation lie not in its use as a calculational tool,” as has been pointed out in ref.[1], “but rather in the deeper insight it affords into the formal structure of mechanics.” We have a greater freedom to select the physical quantities to designate as “coordinates” and “momenta”, which have equal status as independent variables. As a result we can present the physical content of mechanics in a newer and more abstract way. The more abstract formulation is primarily of interest to us today because of its essential role in constructing the framework of modern physics, such as statistical mechanics and quantum theory.

Canonical equations of Hamilton with symmetry

The Newton’s second law of motion in classical mechanics, based on which the Hamiltonian formulation is established, is the second-order differential equation about the evolution coordinate (here the evolution coordinate is time), which is in this paper referred as the second-order differential system. Similarly, the first-order differential system is the first-order differential equation about the evolution coordinate. The Lagrangian density of the second-order differential system of the continuous systems is expressed in general as [1]

$$l = \sum_{s=1}^N \sum_{k=1}^N A_{sk} \dot{q}_s \dot{q}_k + \sum_{s=1}^N B_s \dot{q}_s + C, \quad (8)$$

where A_{sk}, B_s, C depend on not only q_s but also $q_{s,x}$ in general. The generalized momentum can be obtained by the definition (5) as

$$p_s = \sum_{k=1}^N (A_{sk} + A_{ks}) \dot{q}_k + B_s, \quad (9)$$

which is a function of q_s, \dot{q}_s and $q_{s,x}$. The number of Eqs. (9) is N , and there are $4N$ variables, which are q_s, \dot{q}_s, p_s and $q_{s,x}$. So the degree of freedom of Eqs. (9) is $3N$. Then q_s, p_s and

$q_{s,x}$ are taken as independent variables. The generalized velocities \dot{q}_s can be expressed with these independent variables.

Besides the second-order differential systems, there are a number of the first-order differential systems to model physical phenomena, for example, the nonlinear Schrödinger equation (NLSE)

$$i\frac{\partial\varphi}{\partial t} + \frac{1}{2}\frac{\partial^2\varphi}{\partial x^2} + |\varphi|^2\varphi = 0, \quad (10)$$

which is a universal model that describes many nonlinear physical systems and can be applied to hydrodynamics [2], nonlinear optics [3–5], nonlinear acoustics [6], Bose-Einstein condensates [7], and so on. In nonlinear optics [3–5], the NLSE (10) governs the propagation of the slowly-varying light-envelope, where the evolution coordinate t is the propagation direction coordinate, the light-envelope φ is a cw paraxial beam in a planar waveguide [3] or a narrow spectral-width pulse in optical fibers [4, 5], and x is a transverse space coordinate for the beam and a frame moving at the group velocity (the so-called retarded frame) for the pulse, respectively. We will show in the paper that the canonical equations for the first-order differential system are significantly different from Eqs.(3) and (4), which are the canonical equations only for the second-order differential system. But in ref.[1], the canonical equations (3) and (4) were obtained regardless whether the continuous system is the first-order differential system or the second-order differential system. In ref.[4], the canonical equations for the NLSE were considered to be the same as Eqs.(3) and (4). But obviously the canonical equations (3) and (4) are not valid for the NLSE, from which it is impossible to obtain the NLSE or its complex-conjugate equation, as will be shown in the following. Attempt was made to deal with the difficulty in ref.[8], but the canonical equations for the NLSE they obtained were inconsistent, as will be shown in the following. In this paper we obtain the canonical equations with the complete symmetry, which are valid not only for the first-order differential system, but also for the second-order differential system.

For the first-order differential system of the continuous systems, the Lagrangian density must be the linear function of the generalized velocities \dot{q}_s . If the Lagrangian density is a quadratic function of the generalized velocities like Eq.(8), the equation of motion, i.e., the Euler-Lagrange equation

$$\frac{\partial}{\partial t}\frac{\partial l}{\partial \dot{q}_s} - \frac{\delta l}{\delta q_s} = 0 \quad (11)$$

will be the second-order differential equation about the evolution coordinate t , which is

in contradiction with the definition of the first-order differential system. Therefore, the Lagrangian density of the first-order differential system can only be expressed as

$$l = \sum_{s=1}^N R_s(q_s) \dot{q}_s + Q(q_s, q_{s,x}). \quad (12)$$

Besides, R_s in Eq.(12) is not the function of a set of $q_{s,x}$. If R_s is also the function of a set of $q_{s,x}$, there will be such terms as $q_{s,x} \dot{q}_s$ appearing in Eq.(12). Substitution of Eq.(12) into Eq.(11) leads to the appearance of the mixed partial derivative term $\frac{\partial^2 q_s}{\partial x \partial t}$, the term can be changed to $\frac{\partial^2 q_s}{\partial t^2}$ by the rotation of the coordinate frame. After the coordinate transformation, the Euler-Lagrange equation (11) is changed to the second-order partial differential equation of the standard form. Then the system expressed with the standard form is a second-order differential system. Consequently, the generalized momentum p_s , which is obtained by the definition (5) as

$$p_s = R_s(q_s), \quad (13)$$

is only a function of q_s . This is of significant difference from the case of the second-order differential system, where the generalized momentum p_s is the function of not only q_s , but also \dot{q}_s and $q_{s,x}$, as shown in Eq.(9). There are $2N$ variables, q_s and p_s , in Eqs. (13). And the number of Eqs. (13) is N , which also means there exist N constraints between q_s and p_s . So the degree of freedom of the system given by Eqs. (13) is N . Without loss of generality, we take q_1, \dots, q_ν and p_1, \dots, p_μ as the independent variables, where $\nu + \mu = N$. The remaining generalized coordinates and generalized momenta can be expressed with these independent variables as $q_\alpha = f_\alpha(q_1, \dots, q_\nu, p_1, \dots, p_\mu)$ ($\alpha = \nu + 1, \dots, N$), and $p_\beta = g_\beta(q_1, \dots, q_\nu, p_1, \dots, p_\mu)$ ($\beta = \mu + 1, \dots, N$).

We now derive the canonical equations for the first-order differential system. The total differential of the Hamiltonian density h can be obtained by using Eq.(6)

$$\begin{aligned} dh = & \sum_{s=1}^N p_s d\dot{q}_s + \sum_{\eta=1}^{\mu} \dot{q}_\eta dp_\eta \\ & + \sum_{\beta=\mu+1}^N \dot{q}_\beta \left(\sum_{\lambda=1}^{\nu} \frac{\partial g_\beta}{\partial q_\lambda} dq_\lambda + \sum_{\eta=1}^{\mu} \frac{\partial g_\beta}{\partial p_\eta} dp_\eta \right) - dl, \end{aligned} \quad (14)$$

while the total differential of the Lagrangian density $l(q_s, \dot{q}_s, q_{s,x})$ with respect to its argu-

ments is

$$\begin{aligned}
dl = & \sum_{\lambda=1}^{\nu} \frac{\partial l}{\partial q_{\lambda}} dq_{\lambda} + \sum_{\alpha=\nu+1}^N \frac{\partial l}{\partial q_{\alpha}} \left(\sum_{\lambda=1}^{\nu} \frac{\partial f_{\alpha}}{\partial q_{\lambda}} dq_{\lambda} + \sum_{\eta=1}^{\mu} \frac{\partial f_{\alpha}}{\partial p_{\eta}} dp_{\eta} \right) \\
& + \sum_{s=1}^N \frac{\partial l}{\partial \dot{q}_s} d\dot{q}_s + \sum_{s=1}^N \frac{\partial l}{\partial q_{s,x}} dq_{s,x}.
\end{aligned} \tag{15}$$

Substitution of Eq.(15) into Eq.(14) yields

$$\begin{aligned}
dh = & \sum_{\lambda=1}^{\nu} \left(\sum_{\beta=\mu+1}^N \dot{q}_{\beta} \frac{\partial g_{\beta}}{\partial q_{\lambda}} - \sum_{\alpha=\nu+1}^N \frac{\partial l}{\partial q_{\alpha}} \frac{\partial f_{\alpha}}{\partial q_{\lambda}} - \frac{\partial l}{\partial q_{\lambda}} \right) dq_{\lambda} \\
& + \sum_{\eta=1}^{\mu} \left(\dot{q}_{\eta} + \sum_{\beta=\mu+1}^N \dot{q}_{\beta} \frac{\partial g_{\beta}}{\partial p_{\eta}} - \sum_{\alpha=\nu+1}^N \frac{\partial l}{\partial q_{\alpha}} \frac{\partial f_{\alpha}}{\partial p_{\eta}} \right) dp_{\eta} \\
& - \sum_{s=1}^N \frac{\partial l}{\partial q_{s,x}} dq_{s,x}
\end{aligned} \tag{16}$$

Since the total differential of $h(q_1, \dots, q_{\nu}, p_1, \dots, p_{\mu}, q_{s,x})$ with respect to its arguments can be written as $dh = \sum_{\lambda=1}^{\nu} \frac{\partial h}{\partial q_{\lambda}} dq_{\lambda} + \sum_{\eta=1}^{\mu} \frac{\partial h}{\partial p_{\eta}} dp_{\eta} + \sum_{s=1}^N \frac{\partial h}{\partial q_{s,x}} dq_{s,x}$, by comparing this equation with Eq.(16), we obtain $2N$ equations

$$\frac{\partial h}{\partial q_{\lambda}} = \sum_{\beta=\mu+1}^N \dot{q}_{\beta} \frac{\partial g_{\beta}}{\partial q_{\lambda}} - \sum_{\alpha=\nu+1}^N \frac{\partial l}{\partial q_{\alpha}} \frac{\partial f_{\alpha}}{\partial q_{\lambda}} - \frac{\partial l}{\partial q_{\lambda}}, \tag{17}$$

$$\frac{\partial h}{\partial p_{\eta}} = \dot{q}_{\eta} + \sum_{\beta=\mu+1}^N \dot{q}_{\beta} \frac{\partial g_{\beta}}{\partial p_{\eta}} - \sum_{\alpha=\nu+1}^N \frac{\partial l}{\partial q_{\alpha}} \frac{\partial f_{\alpha}}{\partial p_{\eta}}, \tag{18}$$

$$\frac{\partial h}{\partial q_{s,x}} = - \frac{\partial l}{\partial q_{s,x}}, \tag{19}$$

where $\lambda = 1, \dots, \nu, \eta = 1, \dots, \mu, s = 1, \dots, N$. From the Euler-Lagrange equations (11), we obtain

$$\frac{\partial l}{\partial q_s} = \frac{\partial}{\partial t} \frac{\partial l}{\partial \dot{q}_s} + \frac{\partial}{\partial x} \frac{\partial l}{\partial q_{s,x}}. \tag{20}$$

Substituting Eq.(20) into Eqs.(17) and (18), we have

$$\begin{aligned} \frac{\partial h}{\partial q_\lambda} = & -\dot{p}_\lambda + \sum_{\beta=\mu+1}^N \dot{q}_\beta \frac{\partial g_\beta}{\partial q_\lambda} - \sum_{\alpha=\nu+1}^N \dot{p}_\alpha \frac{\partial f_\alpha}{\partial q_\lambda} \\ & - \sum_{\alpha=\nu+1}^N \frac{\partial}{\partial x} \frac{\partial l}{\partial q_{\alpha,x}} \frac{\partial f_\alpha}{\partial q_\lambda} - \frac{\partial}{\partial x} \frac{\partial l}{\partial q_{\lambda,x}}, \end{aligned} \quad (21)$$

$$\begin{aligned} \frac{\partial h}{\partial p_\eta} = & \dot{q}_\eta + \sum_{\beta=\mu+1}^N \dot{q}_\beta \frac{\partial g_\beta}{\partial p_\eta} - \sum_{\alpha=\nu+1}^N \dot{p}_\alpha \frac{\partial f_\alpha}{\partial p_\eta} \\ & - \sum_{\alpha=\nu+1}^N \frac{\partial}{\partial x} \frac{\partial l}{\partial q_{\alpha,x}} \frac{\partial f_\alpha}{\partial p_\eta}, \end{aligned} \quad (22)$$

Then substituting Eq.(19) into Eqs.(21) and (22), we obtain the N canonical equations for the first-order differential system

$$\begin{aligned} \frac{\delta h}{\delta q_\lambda} = & -\dot{p}_\lambda + \sum_{\beta=\mu+1}^N \dot{q}_\beta \frac{\partial g_\beta}{\partial q_\lambda} - \sum_{\alpha=\nu+1}^N \dot{p}_\alpha \frac{\partial f_\alpha}{\partial q_\lambda} \\ & + \sum_{\alpha=\nu+1}^N \frac{\partial}{\partial x} \frac{\partial h}{\partial q_{\alpha,x}} \frac{\partial f_\alpha}{\partial q_\lambda}, \end{aligned} \quad (23)$$

$$\begin{aligned} \frac{\delta h}{\delta p_\eta} = & \dot{q}_\eta + \sum_{\beta=\mu+1}^N \dot{q}_\beta \frac{\partial g_\beta}{\partial p_\eta} - \sum_{\alpha=\nu+1}^N \dot{p}_\alpha \frac{\partial f_\alpha}{\partial p_\eta} \\ & + \sum_{\alpha=\nu+1}^N \frac{\partial}{\partial x} \frac{\partial h}{\partial q_{\alpha,x}} \frac{\partial f_\alpha}{\partial p_\eta}. \end{aligned} \quad (24)$$

To obtain Eq.(24), we have used $\frac{\delta h}{\delta p_\eta} = \frac{\partial h}{\partial p_\eta}$, because h is not a function of $p_{\eta,x}$. The canonical equations (23) and (24) can be expressed in a completely symmetric form as

$$\frac{\delta h}{\delta q_\lambda} = \sum_{s=1}^N \left(\dot{q}_s \frac{\partial p_s}{\partial q_\lambda} - \dot{p}_s \frac{\partial q_s}{\partial q_\lambda} \right) + \sum_{\alpha=\nu+1}^N \frac{\partial}{\partial x} \frac{\partial h}{\partial q_{\alpha,x}} \frac{\partial f_\alpha}{\partial q_\lambda}, \quad (25)$$

$$\frac{\delta h}{\delta p_\eta} = \sum_{s=1}^N \left(\dot{q}_s \frac{\partial p_s}{\partial p_\eta} - \dot{p}_s \frac{\partial q_s}{\partial p_\eta} \right) + \sum_{\alpha=\nu+1}^N \frac{\partial}{\partial x} \frac{\partial h}{\partial q_{\alpha,x}} \frac{\partial f_\alpha}{\partial p_\eta} \quad (26)$$

($\lambda = 1, \dots, \nu$, $\eta = 1, \dots, \mu$, and $\nu + \mu = N$), because $\dot{p}_\lambda = \sum_{\lambda'=1}^\nu \dot{p}_{\lambda'} \frac{\partial q_{\lambda'}}{\partial q_\lambda}$, $\dot{q}_\eta = \sum_{\eta'=1}^\mu \dot{q}_{\eta'} \frac{\partial p_{\eta'}}{\partial p_\eta}$, $\sum_{\eta=1}^\mu \dot{q}_\eta \frac{\partial p_\eta}{\partial q_\lambda} = 0$, and $\sum_{\lambda=1}^\nu \dot{p}_\lambda \frac{\partial q_\lambda}{\partial p_\eta} = 0$. The canonical equations (25) and (26) can be easily extended to the discrete system, which can be expressed as

$$\frac{\partial h}{\partial q_\lambda} = \sum_{s=1}^N \left(\dot{q}_s \frac{\partial p_s}{\partial q_\lambda} - \dot{p}_s \frac{\partial q_s}{\partial q_\lambda} \right), \quad (27)$$

$$\frac{\partial h}{\partial p_\eta} = \sum_{s=1}^N \left(\dot{q}_s \frac{\partial p_s}{\partial p_\eta} - \dot{p}_s \frac{\partial q_s}{\partial p_\eta} \right), \quad (28)$$

where $\lambda = 1, \dots, \nu$, $\eta = 1, \dots, \mu$, and $\nu + \mu = N$.

Although they are derived from the first-order differential system, the canonical equations of Hamilton (25) and (26) with symmetry can be proved to be reduced to Eqs.(3) and (4) without symmetry, if all the N generalized coordinates q_s and N generalized momenta p_s in Eqs.(25) and (26) are independent, because $\frac{\partial g_\beta}{\partial q_\lambda} = \frac{\partial f_\alpha}{\partial q_\lambda} = \frac{\partial g_\beta}{\partial p_\eta} = \frac{\partial f_\alpha}{\partial p_\eta} = 0$ ($\alpha = \nu + 1, \dots, N, \beta = \mu + 1, \dots, N$) in this case. In fact, this is just the case of the second-order differential system. So the canonical equations (25) and (26) with the completely symmetry are valid not only for the first-order differential system, but also for the second-order differential system. The conventional form of the canonical equations (3) and (4) without the symmetry are only for the second-order differential system. In addition, the number of Eqs. (25) and (26) is N in the case for the first-order differential system, and is half of that of Eqs.(3) and (4). This can be explained in the following way. For the second-order differential system, the Euler-Lagrange equations (11) are the second-order differential equations about the evolution coordinate, where the Lagrangian density is expressed as Eq.(8). The number of Eqs.(11) is N . It is well known that one second-order differential equation can be reduced to two first-order differential equations [9]. Then $2N$ canonical equations, which are the first-order differential equations about the evolution coordinate, can be obtained from the N Euler-Lagrange equations. But it is significantly different for the first-order differential system that the Euler-Lagrange equations (11) are the first-order differential equations about the evolution coordinate, so only N canonical equations are obtained from Eqs.(11).

Hamiltonian formulation for the NLSE

After constructing the canonical equations of Hamilton for the first-order differential system, we discuss its application to the NLSE. It is known that the Lagrangian density for the NLSE can be expressed as [10] $l = -\frac{i}{2}(\varphi^* \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \varphi^*}{\partial t}) + \frac{1}{2} |\frac{\partial \varphi}{\partial x}|^2 - \frac{1}{2} |\varphi|^4$. The NLSE is complex, and therefore it is an equation with two real functions, the real part of φ and its imaginary part. It is convenient to consider instead the fields φ and φ^* which are treated as independent from each other. Therefore, N (the components of the quantity) for the NLSE equals two, i.e., there are two generalized coordinates, $q_1 = \varphi^*$ and $q_2 = \varphi$, and two

generalized momenta can be obtained by using the definition (5) as

$$p_1 = \frac{i}{2}\varphi, p_2 = -\frac{i}{2}\varphi^*. \quad (29)$$

The Hamiltonian density for the NLSE can be obtained by use of Eq.(6) as [4]

$$h = -\frac{1}{2} \left| \frac{\partial \varphi}{\partial x} \right|^2 + \frac{1}{2} |\varphi|^4. \quad (30)$$

If the generalized coordinate q_1 and the generalized momentum p_1 are taken as the independent variables, the remaining generalized coordinate q_2 and the remaining generalized momentum p_2 can be expressed via the relations (29) as $q_2 = -2ip_1$ and $p_2 = -\frac{i}{2}q_1$, respectively. We should also note that the Hamiltonian density h is also the function of $q_{s,x}$, which are independent from q_s and p_s . Then for the NLSE, the Hamiltonian density (30) should be expressed with the independent variables $q_1, p_1, q_{1,x}$ and $q_{2,x}$ as

$$h = -\frac{1}{2} q_{1,x} q_{2,x} - 2q_1^2 p_1^2. \quad (31)$$

We should note that $\nu = \mu = 1$ and $N = 2$ for the NLSE, which means that the equations (25) have only one equation, so do Eqs.(26). Therefore, the canonical equations (25) and (26) will yield two equations for the NLSE. The left side of Eq.(25) is obtained as

$$\frac{\delta h}{\delta q_1} = \frac{1}{2} \frac{\partial^2 \varphi}{\partial x^2} + |\varphi|^2 \varphi, \quad (32)$$

and its right side is

$$-\dot{p}_1 + \dot{q}_2 \frac{\partial p_2}{\partial q_1} = -i\dot{\varphi}. \quad (33)$$

Then the NLSE (10) can be obtained. Using the other canonical equation, the left side of Eq.(26) is obtained as

$$\frac{\delta h}{\delta p_2} = -4q_1^2 p_1 = -2i|\varphi|^2 \varphi^*, \quad (34)$$

and its right side is

$$\dot{q}_1 - \dot{p}_2 \frac{\partial q_2}{\partial p_1} + \frac{\partial}{\partial x} \frac{\partial h}{\partial q_{2,x}} \frac{\partial q_2}{\partial p_1} = 2\dot{\varphi}^* + i \frac{\partial^2 \varphi^*}{\partial x^2}. \quad (35)$$

Then the complex conjugate of the NLSE is also obtained. Therefore, the canonical equations (25) and (26) are consistent in the sense that the NLSE can be expressed with one of the two canonical equations, and its complex conjugate can be expressed with the other.

In ref.[4], the canonical equations for the NLSE were considered to be the same as those for the second-order differential system. Then, according to the definition (5), $p_\varphi = \partial l / \partial \dot{\varphi}$,

p_φ must be $-i/2\varphi^*$ but not $-i\varphi^*$. It was artificially doubled in ref.[4] so that $p_\varphi = -i\varphi^*$ in Eq.[5.1.29] (to avoid confusion, we replace the parentheses by the brackets to represent the formulas in the references) to make the NLSE derived from the canonical equation (4). In fact, substitution of the Hamiltonian density (30) into Eq.(4) only yields $\frac{i}{2}\frac{\partial\varphi}{\partial t} + \frac{1}{2}\nabla_\perp^2\varphi + |\varphi|^2\varphi = 0$, which in fact does not be the NLSE (10). In ref.[8], the canonical equations the authors obtained are Eqs.[3.87] and [3.81], the latter is the same as Eq.(7). Although the NLSE can be derived from Eq.[3.87], its complex conjugation could not be obtained from the other, Eq. [3.81]. We now show this claim. Substituting the Hamiltonian density (30) into Eq.(7), the left side of it is obtained as $\frac{\partial\varphi^*}{\partial t}$, and the right side is $\frac{\partial h}{\partial p_{\varphi^*}} = \frac{\partial h}{\partial\varphi} \frac{\partial\varphi}{\partial p_{\varphi^*}} = -2i|\varphi|^2\varphi^*$, where $p_{\varphi^*} = \frac{\partial l}{\partial\varphi^*}$. Then the equation $-\frac{i}{2}\frac{\partial\varphi^*}{\partial t} + |\varphi|^2\varphi^* = 0$ can be obtained, which is absolutely not the complex conjugate of the NLSE. Therefore, the canonical equations for the NLSE obtained in ref.[8] are inconsistent.

Conclusion

Summarizing, we obtain a new form of canonical equations of Hamilton with the complete symmetry, which are valid not only for the first-order differential system, but also for the second-order differential system. The conventional form of the canonical equations without the symmetry are only for the second-order differential system. The number of the canonical equations for the first-order differential system is N rather than $2N$ like the case for the second-order differential system. The canonical equations for the NLSE are two equations, which are consistent in the sense that the NLSE can be expressed with one of the canonical equations and its complex conjugate can be expressed with the other. The Hamiltonian formulation for the continuous system can also be extended to the discrete system.

It is well known that the symmetry plays an important role in theoretical physics [11]. The search for and the discovery of new symmetries promote the exploration of fundamental laws of physics. Based on the idea, the conical equations of Hamilton with the complete symmetry found by us might find their appropriate position in modern theoretical physics.

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